CALCULUS

AP CALCULUS BC

美国大学先修课微积分BC

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第一单元 极限 Chapter 1 Limit

极限的定义、表达与存在性定理 Definition, Expression, and Existence Theorems for Limits

1. 极限的定义与表达形式 Definition and Form of Expression for Limit
   a. 定义 Definition: Let $f(x)$ be a function defined on an interval that contains $x = c$, except possibly at $x = c$. Then we say that,

   $$\lim_{x \to c} f(x) = L$$

   b. 表达形式 Form of Expression

   $$\lim_{x \to c} f(x) = L \quad L\ is\ the\ limit\ of\ f(x) as\ x\ approaches\ c$$

   c. 左极限与右极限 Left Limit and Right Limit
      i. 右极限 Right limit: $\lim_{x \to c+} f(x) = L$ approaching from the right.
      ii. 左极限 Lift limit: $\lim_{x \to c-} f(x) = L$ approaching from the left.
2. 极限的存在性定理 Existence Theorem of Limit
   a. 左极限与右极限相等 Left Limit Is Equal to the Right Limit

   $$\lim_{x \to c+} f(x) = \lim_{x \to c-} f(x) = L$$

   b. 极限不存在的情况 The Case Where the Limit DNE
      i. Left limit is not equal to the right limit.
      ii. As x approaches c from either side, $f(x)$ increases or decreases indefinitely.
      iii. As x approaches c, $f(x)$ oscillates between two fixed values.

极限值的计算 Evaluating Limits Analytically

3. 极限的性质 Properties of Limit
   a. 标量乘法 Scalar Multiplication: We can "factor" a multiplicative constant out of a limit.

   $$scalar\ multiplication: \lim_{x \to c}[bf(x)] = b \lim_{x \to c} f(x)$$

   b. 和/差法则 Sum or Difference: To take the limit of a sum or difference all we need to do is take the limit of the individual parts and then put them back together with the appropriate sign.

$$sum\ or\ difference: \lim_{x\to c}[f(x) \pm g(x)] = \lim_{x\to c} f(x) \pm \lim_{x\to c} g(x)$$

c. 求积法则 Product: We take the limits of products in the same way that we can take the limit of sums or differences. Just take the limit of the pieces and then put them back together. Also, as with sums or differences, this fact is not limited to just two functions.

$$product: \lim_{x\to c}[f(x)g(x)] = \lim_{x\to c} f(x) \lim_{x\to c} g(x)$$

d. 求商法则 Quotient: As noted in the statement we only need to worry about the limit in the denominator being zero when we do the limit of a quotient. If it were zero we would end up with a division by zero error and we need to avoid that.

$$quotient: \lim_{x\to c}\frac{f(x)}{g(x)} = \frac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)}$$

e. 幂次运算 Power: In this property n can be any real number (positive, negative, integer, fraction, irrational, zero, etc.). In the case that n is an integer this rule can be thought of as an extended case of "Product".

$$power: \lim_{x\to c}[f(x)]^n = [\lim_{x\to c} f(x)]^n$$

4. 极限的计算 Calculation of Limits
   a. 直接代入法 Direct Substitution

$$\lim_{x\to c} f(x) = f(c)$$

   b. 不定型极限 Indeterminate Limit
      i. 极限的不定型 Indeterminate Form of Limit: 如果我们使用直接代入法 Direct Substitution 后发现，原极限变成了$0/0, 0\cdot\infty, \infty/\infty$等类型的话，我们称其为不定型 Indeterminate Form。这个时候我们需要用其他方法来求解极限。
      ii. 方法一: 恒等变形法 Method I: Identical Deformation Method
         - 十字相乘 Dividing Out Technique
         - 共轭因式 Conjugate Factor Technique
         - 其他方案恒等变形直到可以直接代入。
      iii. 方法二: 洛必达法则 Method II: L'Hôpital's rule

$$\lim_{x\to c}\frac{f(x)}{g(x)} = \lim_{x\to c}\frac{f'(x)}{g'(x)}$$

   c. 特殊的三角函数极限 Special Trigonometric Limits

$$\lim_{x\to 0}\frac{sinax}{ax} = 1 \qquad \lim_{x\to 0}\frac{sinax}{sinbx} = \frac{a}{b}$$

d. 三明治定理求极限 Squeeze Theorem to Find Limits

$$If\ h(x) \le f(x) \le g(x)\ and\ \lim_{x\to c} h(x) = \lim_{x\to c} g(x) = L, thus, \lim_{x\to c} f(x) = L$$

e. 分段函数求极限 Finding the Limit for a Piecewise Function

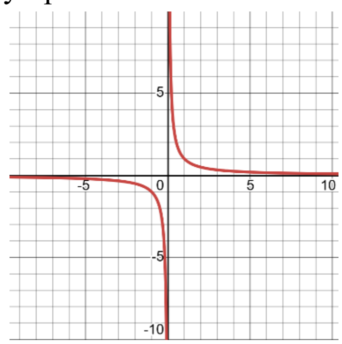
i. 对于分段函数，当我们需要求极限时，我们在段点处求左右极限。

ii. 具体来说，我们单独计算单侧限值，并使用极限的存在性定理 Existence Theorem of Limit 判断其是否存在。

5. 无穷极限与无穷处极限 Infinite Limits and Limits at Infinity

a. 无穷极限 Infinite Limits $\lim_{x\to a+} f(x) = \pm\infty\ \ \lim_{x\to a-} f(x) = \pm\infty$

i. 垂直渐近线 Vertical Asymptote

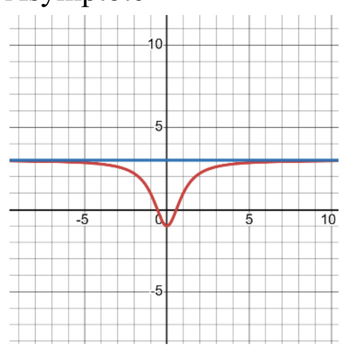


ii. 可能出现的不定型: 1/0 Form

$$解决方案: x \to 1^+\ \ \frac{1}{0^-} = \infty;\ \ x \to 1^-\ \ \frac{1}{0^+} = \infty$$

b. 无穷处极限 Limits at Infinite $\lim_{x\to\infty} f(x) = L\ \lim_{x\to-\infty} f(x) = L$

i. 水平渐近线 Horizontal Asymptote



ii. 可能出现的不定型: ∞/∞ Form

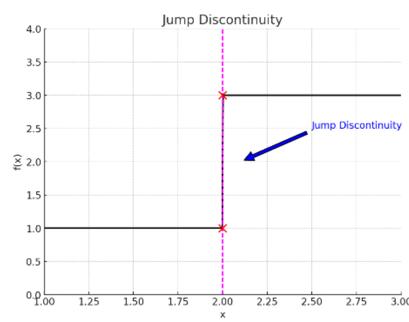
- 方法一: 恒等变形法（分子分母同时除以分母的最高次幂）
- 方法二: 增长速度法（通过把增长速度慢的函数忽略来估算极限值，比如我们应该忽略系数，记录指数对数）

函数的连续性 Continuity of Function

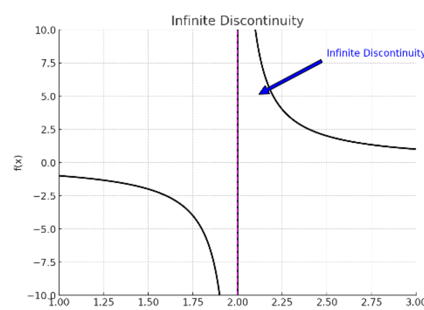
6. 函数的连续性 Continuity of Function
   a. 函数在段点处连续条件 Continuity Condition of the Function at Point: 左极限等于右极限等于函数值

$$\lim_{x\to c+} f(x) = \lim_{x\to c-} f(x) = f(x)$$

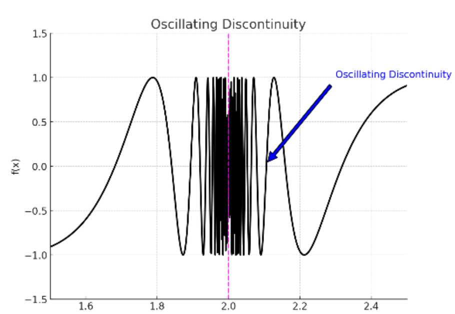
   b. 不连续性的类型 Type of Discontinuity
      i. 可去间断 Removable Discontinuity:是指图形上未定义 Undefined 或与图形其余部分不符的点。具体来说，其极限值存在，函数值不存在或不相等。
      ii. 不可去间断 Unremovable Discontinuity: 函数的极限不存在于某一特定点的不连续类型
         - 跳跃间断点 Jump Discontinuity



         - 无限间断点 Infinite Discontinuity

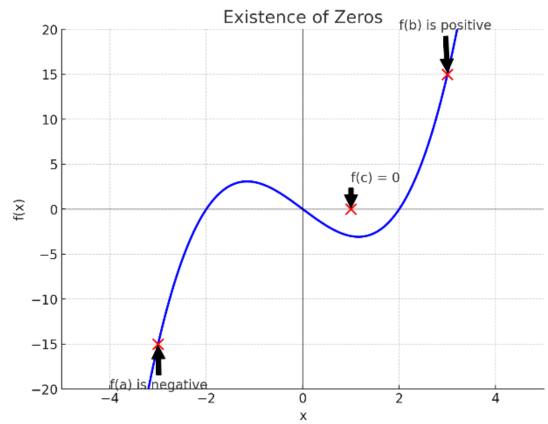


         - 震荡间断点 Oscillating Discontinuity

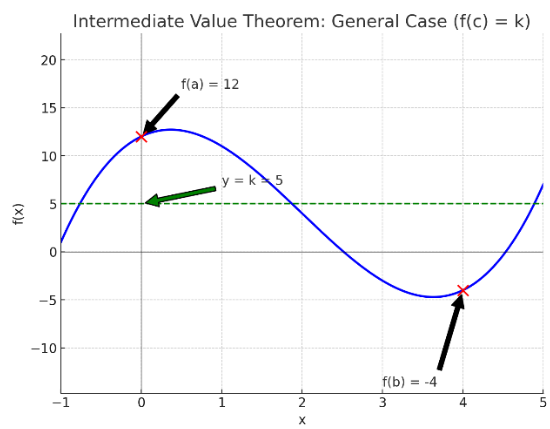


7. 连续区间与不连续区间 Continuous and Discontinuous Intervals
    a. 连续区间 Continuous Interval: 即定义域，求解定义域即可
    b. 不连续区间 Discontinuous Interval: 不在定义域的点
8. 零点存在性定理与介值定理 Existence of Zeros and Intermediate Value Theorem
    a. 零点存在性定理 Existence of Zeros: 若函数两端一正一负，则必有零点出现

$$f(a)f(b)\ opposite - at\ least\ c \in (a,b) for\ which\ f(c) = 0$$



    b. 介值定理 Intermediate Value Theorem (IVT): 介值定理是在连续函数的一个区间内的函数值肯定介于最大值和最小值之间。



第二单元 微分运算与应用 Chapter 2 Differentiation Calculation and Application

导数 Derivative

1. 导数的定义式与导函数 Definition of Derivative and Derivative Function
   a. The derivative of $f\ (x)$ with respect to x is the function $f'(x)$ and is defined as,
   $$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=f'(x)$$
   b. 导数值等于切线斜率 Derivative is equal to slope of tangent line at one point.
2. 可导性与连续性 Differentiability and Continuity
   a. 可导性意味着连续性 Differentiability Implies Continuity
      i. 可导必连续，但连续不一定可导
      ii. 连续不可导点
         - 角点 Corner
         - 尖点 Cusp
         - 竖直切线 Vertical Tangent Line
   b. 可导意味着光滑函数 Smooth Function
      i. 光滑函数 Smooth Function: 在数学中特指无穷可导的函数，不存在尖点 Cusp，也就是说所有的有限阶导数都存在。例如，指数函数就是光滑的，因为指数函数的导数是指数函数本身。

微分运算 Differentiation Operation

3. 基础运算法则 Basic Arithmetic Rules

$$Power\ rule\ \frac{d}{dx}[x^n]=nx^{n-1}$$

$$Product\ rule\ \frac{d}{dx}[f(x)g(x)]=f'(x)g(x)+f(x)g'(x)$$

$$Quotient\ rule\ \frac{d}{dx}\left[\frac{f(x)}{g(x)}\right]=\frac{f'(x)g(x)-f(x)g'(x)}{[g(x)]^2}$$

$$\frac{d}{dx}[cf(x)]=cf'(x)\qquad \frac{d}{dx}[f(x)+g(x)]=f'(x)+g'(x)$$

4. 三角运算法则 Trigonometric Rules

$$\frac{d}{dx}[sinx]=cosx\qquad \frac{d}{dx}[cosx]=-sinx$$

$$\frac{d}{dx}[tanx]=sec^2x\qquad \frac{d}{dx}[cotx]=-csc^2x$$

$$\frac{d}{dx}[secx] = secxtanx \quad \frac{d}{dx}[cscx] = -cscxcotx$$

5. 指对数法则 Exponential and Logarithmic Rules

$$\frac{d}{dx}a^x = a^x lna \quad \frac{d}{dx}e^x = e^x$$

$$\frac{d}{dx}log_a^x = \frac{1}{xlna} \quad \frac{d}{dx}lnx = \frac{1}{x}$$

6. 链式法则 Chain Rule

$$[f(g(x))]' = f'(g(x)) \times g'(x)$$

复合函数求导，需对复合函数先求导，再对复合部分单独求导。

7. 隐函数求导 Implicit differentiation

$$\frac{d}{dx}g(y) = g'(y)\frac{dy}{dx}$$

隐函数求导为链式法则特殊应用，两边求导结束后需加dy/dx。

8. 反函数求导 Inverse Function Differentiation

$$(f^{-1})'(b) = \frac{1}{f'(a)}$$

a为原函数定义域为反函数值域；b为原函数值域为反函数定义域。

微分应用 Differentiation Application

9. 第一中值定理 Mean Value Theorem (MVT)

$f(x)$在$[a,b]$上连续并且在(a,b)上可导，则 c 属于(a,b)

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

10. 洛必达法则 L'Hospital's Rule

若为不定型 Indeterminate forms，则通过分子分母同时求导来求极限。

$$\lim_{x\to c}\frac{f(x)}{g(x)} = \lim_{x\to c}\frac{f'(x)}{g'(x)}$$

11. 最大化问题 Optimization
    a. Step 1: 设变量，写出变量关系式，写定义域。
    b. Step 2: 化为函数式（多变量化为单变量），求因变量最值。
    c. Step 3: 用导数法求最值 Absolute Extrema
        i. 闭区间: Candidate Test
        ii. 开区间: 单调性求最值
            - f changes from increasing to decreasing at c, f is a max.

- f changes from decreasing to increasing at c, f is a min.

12. 相关变化率问题 Related rates
    a. Step 1: 读出题目条件，转化为数学表达式，找出已经有的 rate，用 d_/dt 表示，找出需要求的 rate。
    b. Step 2: 设变量写出变量关系式（变量为已知 rate 导数式分子的变量），找出两个变量的关系。
    c. Step 3: 隐函数求导，得出两个 rate 的关系式。
    d. Step 4: 代值并回答。
13. 线性近似 Linear approximation

    已知$f(x)$, x_0, $x \to x_0$, 需要近似$f(x)$

    $$f(x) = f(x_0) + f'(x_0)(x - x_0)$$

    $(x - x_0)$为未知减去已知

利用导数分析函数 Analyzing Functions Using Derivatives

14. 临界点 Critical Points
    a. Step 1: 对函数$f(x)$求导。
    b. Step 2: 若$f'(x)$为分式，则分子分母同时为 0，求得x的值。若为整式，则整式为 0，求得x的值。

       $f'(x) = 0$ or $f'(c)$ is undefined, but $f(c)$ is defined — critical points
15. 绝对极值 Absolute Extrema
    c. Step 1: find critical points.
    d. Step 2: find endpoints of the domain.
    e. Step 3: candidate test

       把 critical points 和 endpoints 带入函数，求得每个点对应的y值。最大y值为 absolute maximum，最小y值为 absolute minimum。
16. 增减性 Increasing or Decreasing
    f. Step 1: Draw the image of the first-order derivative function. 穿针引线
    g. Step 2: Judged by the interval positive and negative. 根据区间正负判断

       $f'(x) > 0$ — $f(x)$ is increasing on (a, b)

       $f'(x) < 0$ — $f(x)$ is decreasing on (a, b)
17. 局部极值 Local Extrema

    关键特征：函数增减改变之处；一阶导正负改变之处。
    h. Method 1: First Derivative Test
        i. Step 1: Find critical points c

ii. Step 2: Draw the image of the first-order derivative function.穿针引线

iii. Step 3: Judging from the trend of positive and negative image changes.

$f'(x)$ changes from positive to negative at c — $f(c)$ is a local maximum.

$f'(x)$ changes from negative to positive at c — $f(c)$ is a local minimum.

i. Method 2: Second Derivative Test

i. $f'(c) = 0, f''(c) > 0$ — $f(c)$ is a local minimum.

ii. $f'(c) = 0, f''(c) < 0$ — $f(c)$ is a local maximum.

iii. $f'(c) = 0, f''(c) = 0$ — test fails and use the First Derivative Test.

18. 凹性 Concavity

j. 分类 Classification

i. Concave up: U 朝上，曲线位于曲线上任意一点处的切线上方。

concave up on the Interval if $f'(x)$ is increasing on I — $f''(x) > 0$ on I

ii. Concave down: U 朝下，曲线位于曲线上任意一点处的切线下方。

concave down on the interval if $f'(x)$ is decreasing on I — $f''(x) < 0$ on I

k. 确定函数的凹度 The Concavity of the Function

i. Step 1: Draw the image of the second-order derivative function. 穿针引线

ii. Step 2: Judged by the interval positive and negative. 根据区间正负判断

$f''(x)$ is positive on this interval — $f(x)$ is concave up

$f''(x)$ is negative on this interval — $f(x)$ is concave down

19. 拐点 Inflection Points

关键特征：函数凹性改变之处; 一阶导增减改变之处; 二阶导正负改变之处。

l. Step 1: $f''(x) = 0$ or $f''(c)$ is undefined but $f(c)$ is defined

m. Step 2: Draw the image of the second-order derivative function. 穿针引线

n. Step 3: Judging from the trend of positive and negative image changes. 根据图像正负变化趋势判断

$f''(x)$ changes from positive to negative at c — $f(x)$ has an inflection point at $x = c$

$f''(x)$ changes from negative to positive at c — $f(x)$ has an inflection point at $x = c$

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)}$$

第三单元 积分运算与应用 Chapter 3 Integration Calculation and Application

不定积分计算 Indefinite Integral Calculation

1. 不定积分的表达 Representation of Indefinite Integration
   a. 不定积分是一个函数，它是另一个函数的反导 Antiderivative。它可以直观地表示为一个积分符号、一个函数以及末尾的dx。
   b. 不定积分的数学语言 Mathematical Representation of Indefinite Integration

$$y = \int f(x)dx = F(x) + c$$

   $f(x)$为被积函数 integrand；$F(x)$为$f(x)$的反导 antiderivative；c为积分常数 constant of integration。

2. 基础不定积分公式 Basic Integration Rules
   a. 基础运算法则 Basic Arithmetic Rules

$$\int 0\,dx = c \quad \int kdx = kx + c \quad \int kf(x)dx = k\int f(x)dx$$

   b. 幂运算 Power Rules

$$\int x^n dx = \frac{x^{n+1}}{n+1} + c \quad \int \frac{1}{x} = \ln|x| + c$$

   c. 三角运算 Trigonometric Rules

$$\int sindx = -cosx + c \quad \int cosdx = sinx + c$$

$$\int sec^2dx = tanx + c \quad \int csc^2dx = -cotx + c$$

$$\int secx\ tanxdx = secx + c \quad \int cscx\ cotxdx = -cscx + c$$

   d. 反三角运算 Inverse Trigonometric Rules

$$\int \frac{1}{\sqrt{1-x^2}}dx = arcsinx + c$$

$$\int \frac{1}{1+x^2}dx = arctanx + c$$

$$\int \frac{1}{|x|\sqrt{x^2-1}}dx = arcsecx + c$$

   e. 指数运算 Exponential Rules

$$\int e^x dx = e^x + c \quad \int a^x dx = \frac{a^x}{lna} + c$$

3. U 代换解不定积分 U-Substitution to Solve Indefinite Integrals
   a. 条件 Condition: 被积函数可以写成“复合（复合函数）× 内导（复合部分的导

数）”的形式。

b. 步骤 Steps

   i. Step 1: 设复合函数的复合部分为u，写出du（u的导数）。

   ii. Step 2: 把u与du代入原方程，du可以写成“系数×du”的形式。原方程变为对u的积分。

   iii. Step 3: 算出积分答案（代u的式子）。

   iv. Step 4: 把u带入答案，得出结果。

c. 一些经常出现的算式 Some Frequently Occurring Equations

$$\int \frac{f'(x)}{f(x)} dx = ln|f(x)| + c$$

$$\int f(ax+b)dx = \frac{1}{a}F(ax+b) + c$$

d. 案例 Example

   i. Solve the following indefinite integral.

$$\int (2x-1)^5 dx$$

   ii. Step 1: Let $u = 2x - 1$, then $du/dx = 2$

   iii. Step 2: Slove it and we can get $du/2 = dx$. The original indefinite integral will be

$$\int (2x-1)^5 dx = \int u^5 \frac{du}{2} = \frac{1}{2}\int u^5 du$$

   iv. Step 3: Using the basic indefinite integral calculation rule, we can get

$$\frac{1}{2}\int u^5 du = \frac{1}{12}u^6 + c$$

   v. Step 4: The final result will be

$$\frac{1}{12}u^6 + c = \frac{1}{12}(2x-1)^6 + c$$

4. 部分分式解不定积分 Partial Fractions to Solve Indefinite Integrals

a. 条件 Conditions

   i. 分母要能分解成几个不同的线性表达式。

   ii. 分子最高次幂为小于分母最高次幂，即被积函数为真分式。

b. 步骤 Steps

   i. Step 1: 证明不能用u代换。

   ii. Step 2: 利用十字相乘法分解被积函数的分母。

   iii. Step 3: 把被积函数分开，原被积函数分母的每个项分别为每个部分分式的分母。每个部分分式的分子设未知数 A, B, C。

   iv. Step 4: 求解 A, B, C。

- 原被积函数等于每个部分分式之和。
- 每项同时乘原被积函数分母（通分）。
- 带入原被积函数分母的根，求出 A, B, C。

v. Step 5: 带入 A, B, C 到原方程，利用不定积分的线性展开求解。

c. 案例 Example

i. Solve the following indefinite integral.

$$\int \frac{x+4}{x^2-x-2}dx$$

ii. Step 1: Since the numerator is not a derivative of the denominator, we cannot use U-substitution.

iii. Step 2: Factorize the denominator of the integrand.

$$\frac{x+4}{x^2-x-2} = \frac{x+4}{(x-2)(x+1)}$$

iv. Step 3: Decompose the faction.

$$\frac{x+4}{(x-2)(x+1)} = \frac{A}{x-2} + \frac{B}{x+1}$$

v. Step 4: To find A and B.

- Multiply both sides to eliminate denominators.

$$x+4 = A(x+1) + B(x-2)$$

- Distribute and combine like terms.

$$x+4 = (A+B)x + (A-2B)$$

- Set coefficients equal and solve the system.

$$1 = A+B \quad 4 = A-2B$$

$$A = 2 \quad B = -1$$

vi. Step 5: Rewrite the original indefinite integral and solve it.

$$\int \frac{x+4}{x^2-x-2}dx = \int \frac{2}{x-2} - \frac{1}{x+1}dx$$

$$\int \frac{2}{x-2} - \frac{1}{x+1}dx = \int \frac{2}{x-2}dx - \int \frac{1}{x+1}dx$$

$$= 2\ln|x-2| - \ln|x+1| + C$$

d. 不符合条件的特殊情况处理方法 Treatment of Ineligible Special Cases

i. 如果出现分子最高次幂大于等于分母最高次幂的情况，可以使用如下两个方法来解决。

ii. 分母为一项: 直接拆开为几个式子，利用不定积分的线性展开求解。

iii. 分母为多项式: 使用长除法 Long Division。

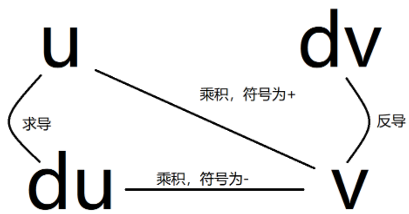
5. 分部积分法 Integration by Parts

a. 条件 Condition: 被积函数必须为“代数函数×超越函数”。

b. 步骤 Steps

  i. Step 1: 通过如下法则设出u与dv。
    - x & x^n与$\ln x$的乘积，设$u = \ln x$。
    - x与$sinx, cosx$的乘积，设$u = x$。
    - x与e^x的乘积，设$u = x$。

  ii. Step 2: 利用如下法则，写出等式。



$$\int udv = uv - \int vdu$$

  iii. Step 3: 运用基础积分法则计算。

c. 案例 Example

  i. Solve the following indefinite integral.

$$\int x^2 e^x dx$$

  ii. Step 1: Set u and dv.

$$u = x^2 \;\; dv = e^x dx$$

  iii. Step 2: Calculate the du and v and setup the formula.

$$du = 2xdx \;\;\; v = e^x$$

$$\int x^2 e^x dx = x^2 e^x - \int e^x 2x\, dx = x^2 e^x - 2\int e^x x\, dx$$

  iv. Step 3: Calculate the indefinite integral we get. (We need to use Integration by Parts to calculate this)

$$u = x \;\; dv = e^x dx \;\; du = dx \;\; v = e^x$$

  v. Step 4: Setup the formula.

$$x^2 e^x - 2\int e^x x\, dx = x^2 e^x - 2(xe^x - \int e^x\, dx)$$

  vi. Step 5: Find the solution.

$$x^2 e^x - 2\left(xe^x - \int e^x\, dx\right) = x^2 e^x - 2xe^x + 2e^x + c$$

d. 列表积分法 Tabular Integration

  i. 列表积分法 Tabular Integration

  形如$\int f(x)g(x)\, dx$的积分，其中$f(x)$可以重复求导直到出现 0，而$g(x)$可以简单重复积分。这个时候我们可以使用分部积分法

Integration by Parts，并且列表计算。

ii. 案例 Example: Evaluate $\int x^2 e^x dx$.

- Let $f(x) = x^2$ and $g(x) = e^x$, we list:

f(x) and its derivatives	g(x) and its integrals	Relation
x^2	e^x	(+)
$2x$	e^x	(−)
2	e^x	(+)
0	e^x	(+)

- Combine the products of the functions connected by the arrows according to the operation signs the arrows to obtain.

$$\int x^2 e^x \, dx = x^2 e^x - 2xe^x + 2e^x + c$$

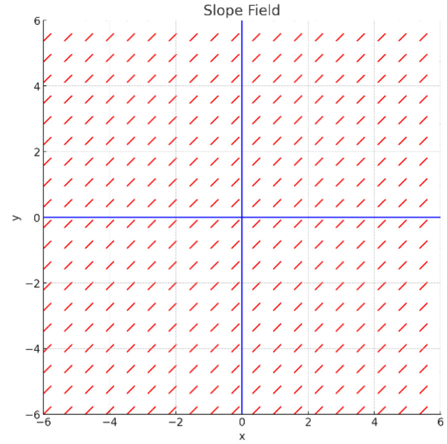
不定积分运用 Indefinite Integration Application

6. 微分方程 Differential Equations
   a. 求解微分方程 Solving Differential Equations
      i. 可分离变量的微分方程 Separable Differential Equation
         - Step 1: 把y'变成dy/dx。
         - Step 2: 标准化成$a/b = c/d$; $ad = bc$。
         - Step 3: 含y项包括dy放左边，含x项包括dx放右边。

$$f(y)dy = f(x)dx$$

         - Step 4: 两边同时求不定积分，只在右边加c得到通解 General Solution。
         - Step 5: 把初值 Initial Condition 代入通解 General Solution。
         - Step 6: 化简后得到特解 Particular Solution。

      ii. 通过斜率场求微分方程 Differential Equations Through the Slope Field
         - 斜率 Slope: 方程$y' = f(x,y)$给了我们在(x,y)平面上每一个点的斜率 Slope。我们可以把$f(x,y)$理解为图中经过点(x,y)的斜率。比如当$f(x,y) = xy$时，点(2,1.5)的斜率是$xy = 2 \times 1.5 = 3$。因此，如果$y(x)$是解且$y(2) = 1.5$，那么$y'(2) = 3$。
         - 斜率场 Slope Field: 在一个平面直角坐标系中，我们可以根据微分方程，对于每一个点的坐标画出其对应的斜率。虽然我们没有办法画出所有点的斜率，但是我们可以选择一些网格点，通过斜率看出这个微分方程的变化趋势。



在每个点上画切线斜率（导数值）

7. 微分方程应用 Differential Equation Application
   a. 放射性元素减少 Exponential Decay

$$y'(t) = ky(t) \quad y(t) = C \times e^{kt}$$

半衰期 Half-life: $y(t_0) = c/2$

   b. 牛顿冷却定律 Newton's Cooling

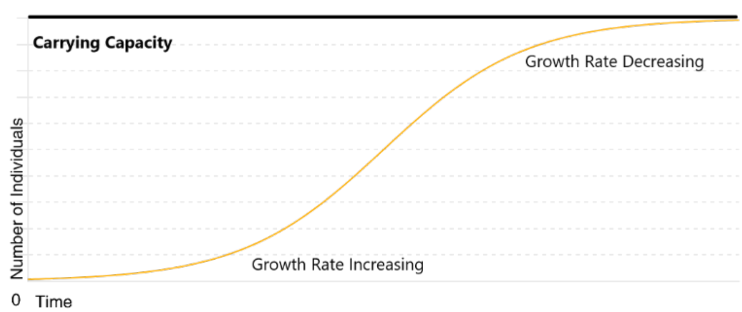
$$y'(t) = k[y(t) - T_a] \quad y(t) = Ae^{kt} + T_a$$

   c. 逻辑斯谛增长 Logistic Growth
      i. 有限制增长 Restricted Growth

$$y'(t) = ky(t)[A - y(t)] \quad \frac{dP}{dt} = kP\left(1 - \frac{P}{K}\right) \quad P = A(t)$$

P: population at time t; K: Limiting or Carrying Capacity (max size of population); k: constant of proportionality

      ii. 图形 Graph



- $y(0) < A$为“S”型
- $y(0) > A$为“捺”型

      iii. 关于 K 值和 P 值的常见问题 FAQ about K-Values and P-Values

- 求 K 值: 把函数标准化后，直接得出 K 值；或者$K = \lim_{t \to \infty} P(t)$
- 求最快增长速率（P 值）: $P = K/2$

8. 使用欧拉方法线性近似 Linear Approximation Using Euler's Method
   a. 对于一个初值问题 Initial-Value Problem (IVP)

   $$y' = f(x, y), y(x_0) = y_0$$

   我们可以使用欧拉方法 Euler's Method 近似该问题的解

   $$x_n = x_0 + nh$$

   $$y_n = y_{n-1} + hf(x_{n-1}, y_{n-1})$$

   这里，$h > 0$ 代表步长 Step Size ，并且n是一个从 1 开始的整数。所走的步数由变量 n 来计算。
   b. 案例 Example
      i. Consider the initial-value problem

      $$y' = 3x^2 - y^2 + 1, y(0) = 2$$

      Use Euler's method with a step size of 0.1 to generate a table of values for the solution for values of x between 0 and 1.
      ii. Solution: We are given $h = 0.1$ and $f(x, y) = 3x^2 - y^2 + 1$. Furthermore, the initial condition $y\,(0) = 2$ gives $x_0 = 0$ and $y_0 = 2$. Using Euler's method with $n = 0$, we can generate the following result.
         - $n = 0$ $x_n = 0$, we can find that $y_n = 2$
         - $n = 1$ $x_n = 0.1$, we can find that $y_n = 1.7$
         - $n = 2$ $x_n = 0.2$, we can find that $y_n = 1.514$
         - $n = 3$ $x_n = 0.3$, we can find that $y_n = 1.3968$
         - $n = 4$ $x_n = 0.4$, we can find that $y_n = 1.3287$
         - $n = 5$ $x_n = 0.5$, we can find that $y_n = 1.3001$
         - $n = 6$ $x_n = 0.6$, we can find that $y_n = 1.3061$
         - $n = 7$ $x_n = 0.7$, we can find that $y_n = 1.3435$
         - $n = 8$ $x_n = 0.8$, we can find that $y_n = 1.4100$
         - $n = 9$ $x_n = 0.9$, we can find that $y_n = 1.5032$
         - $n = 10$ $x_n = 1.0$, we can find that $y_n = 1.6202$

      With ten calculations, we are able to approximate the values of the solution to the initial-value problem for values of x between 0 and 1.

定积分计算 Definite Integral Calculation

9. 黎曼和法求定积分 Riemann Sum Method for Definite Integral
   a. 定积分是黎曼和的极限 Definite Integral Is the Limit of the Riemann Sum.

$$\lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x = \int_a^b f(x)dx$$

b. 四步确定定积分 Four Steps to Determine the Definite Integral
   i. 分割 Partition
   ii. 近似构造 Construction
   iii. 求和 Summation
   iv. 精确值（极限值）Limitation

c. 原理: 极限与黎曼和 Limits and Reimann Sums
   i. 早先，曲线下的面积是根据和的极限 Limit of Sums 来定义的

$$A = \lim_{n\to\infty} S(P) = \lim_{n\to\infty} T(P)$$

其中，S(P)和 T(P)分别是

$$S(P) = \sum_{1}^{n} m_i(x_i - x_{i-1}) = m_1(x_1 - x_0) + m_2(x_2 - x_1) + \cdots$$

$$T(P) = \sum_{1}^{n} M_i(x_i - x_{i-1}) = M_1(x_1 - x_0) + M_2(x_2 - x_1) + \cdots$$

S(P)和 T(P)就是黎曼和 Reimann Sums 的例子之一。

   ii. 黎曼和的形式 Form of Reimann Sums 是$\sum_{i=1}^{n} f(x_i)\Delta x$，其中每一个$c_i$都是我们用来求第 i 个子区间内矩形长度的值。由于函数是连续的，我们可以使用子区间内的任意点来求取极限。为了利用极限的概念，我们让每个矩形的宽度趋近于 0，这相当于让矩形的数量 n 趋近于无穷大。这样，我们就能找到曲线下的精确面积。

$$\lim_{n\to\infty} A_n = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x$$

   iii. 一般情况的定义 Definition of the General Situation
      - f is continuous on $[a, b]$
      - The interval $[a, b]$ is divided into n sub-intervals of equal width Δx, with $\Delta x = \frac{b-a}{n}$.
      - The endpoints of these sub-intervals are $x_0 = a, x_1, x_2, \ldots, x_n = b$.
      - $x_1^*, x_2^*, \ldots, x_n^*$ are any sample points in these sub-intervals, then the definite integral of f from $x = a$ to $x = b$ is

$$\lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x = \int_a^b f(x)dx$$

10. 使用黎曼和近似定积分 Approximation of Definite Integrals with Riemann Sum

a. 定积分近似 Definite Integrals Approximation

$$\int_a^b f(x)dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x \approx \sum_{i=1}^{n} f(x_1)\Delta x$$

b. 四种黎曼和近似定积分 Four Types of Approximation of Definite Integral

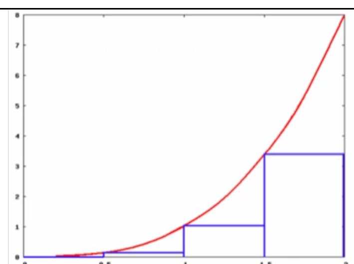
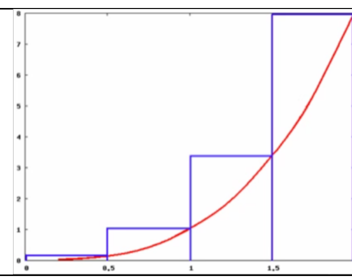
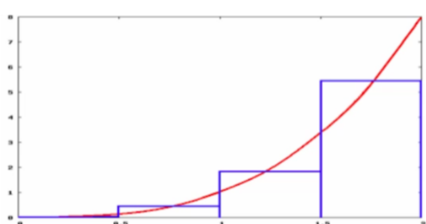
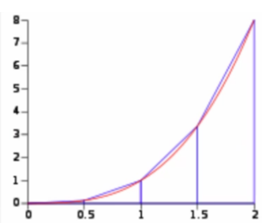
$$R(n) = \sum_{i=1}^{n} f(x_i)\Delta x = f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_n)\Delta x$$

$$L(n) = \sum_{i=1}^{n} f(x_i - 1)\Delta x = f(x_0)\Delta x + f(x_1)\Delta x + \cdots + f(x_n - 1)\Delta x$$

$$M(n) = \sum_{i=1}^{n} f\left(\frac{x_{i-1} + x_i}{2}\right)\Delta x = f\left(\frac{x_0 + x_1}{2}\right)\Delta x + \cdots + f\left(\frac{x_{i-1} - x_i}{2}\right)\Delta x$$

$$T(n) = \frac{b-a}{2n}[f(x_0) + 2f(x_1) + 2(fx_2) + \cdots + 2f(x_n - 1) + f(x_n)]$$

四种黎曼和近似定积分分别是左黎曼和 Left Riemann Sum、右黎曼和 Right Riemann Sum、中点黎曼和 Midpoint Riemann Sum 和梯形黎曼和 Trapezoidal Riemann Sum。

Left Riemann Sum	Right Riemann Sum
	
Midpoint Riemann Sum	**Trapezoidal Riemann Sum**
	

c. 四种近似的比较 Comparison of Four Approximations

i. 单调递增: $L(n)$ <真实值< $R(n)$

ii. 单调递减: $L(n)$ >真实值> $R(n)$

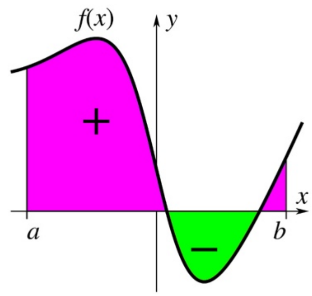
iii. Concave Up: $T(n)$ >真实值> $M(n)$

iv. Concave Down: $T(n)$ <真实值< $M(n)$

11. 图形法求定积分 Graphical Method for Definite Integral

a. 计算方法 Method: x轴上方面积减去x轴下方面积。

b. 图示 Graph



12. 定积分性质 Definite Integral Properties

$$\int_a^a f(x)dx = 0 \quad \int_b^a f(x)dx = -\int_a^b f(x)$$

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

$$\int_a^b kf(x)dx = k\int_a^b f(x)dx \quad \int_a^b [f(x) \pm g(x)]dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$$

13. 微积分基本定理 Fundamental Theorem of Calculus

a. 微积分基本定理求定积分 FTC to Solve Definite Integral

$$\int_a^b f(x)dx = F(b) - F(a) = F(x)|_a^b \quad F'(x) = f(x), F(x) = \int f(x)dx$$

b. 微积分第二基本定理 The Second Fundamental Theorem of Calculus

i. 积分上限函数/累计函数 Accumulation function

$$F(x) = \int_a^x f(t)dt$$

ii. 微积分第二基本定理 The Second Fundamental Theorem of Calculus

$$\frac{d}{dx}\left[\int_a^x f(t)dt\right] = f(x) \quad \frac{d}{dx}\left[\int_a^{g(x)} f(t)dt\right] = f\big(g(x)\big)g'(x)$$

14. 原不定积分方法求解定积分 Solve the Definite Integral Using the Original Indefinite Integral Method

a. U 代换求定积分 U-Substitution to Solve Definite Integrals

$$\int_a^b f(g(x))g'(x)dx = \int_{g(a)}^{g(b)} f(u)du$$

$$u = g(x) \quad du = g'(x)du \quad u(a) = g(a) \quad u(b) = g(b)$$

b. 分部积分法求定积分 Integration by Parts to Solve Definite Integrals
   i. 分部积分法求定积分公式 Formula

$$\int_{x=a}^{x=b} udv = uv|_{x=a}^{x=b} - \int_{x=a}^{x=b} vdu$$

   ii. 除了使用上述公式，也可以用分部积分法把反导彻底求出，再用微积分基本定理 FTC 算对应定积分。

c. 部分分式法求定积分 Practical Fractions to Solve Definite Integrals: 用部分分式法把反导彻底求出，再用微积分基本定理 FTC 算对应定积分。

定积分应用 Definite Integration Application

15. 积分中值定理 Mean Value Theorems for Definite Integrals
If f is continuous on the closed interval $[a, b]$, then there exists a number c in the closed interval $[a, b]$, such that

$$\int_a^b f(x)dx = f(c)(b-a)$$

16. 应用一: 函数的均值 Application 1: The Average Value of the Function
    a. 函数均值公式 Formula to The Average Value of the Function

$$f(c) = \frac{1}{b-a}\int_a^b f(x)dx$$

被积函数为所求均值的函数，$f(c)$为函数的均值。

    b. 函数均值求解的本质是积分中值定理中的$f(c)$。

17. 应用二: 净变化定理 Application 2: Net Change Theorem
    a. 净变化问题 Net Change Problem
       i. 积分手段求解函数改变量问题。是已知 Rate 求 Function 的问题。
       ii. 净变化问题公式 Formula to Net Change Problem

$$\int_a^b f(x)dx = F(b) - F(a)$$

b. 衍生问题 Derived Problem

i. 数学衍生问题 Mathematical Derivation Problem

$$Net\ Change\ of\ F(x)\ during\ [a,b] = \int_a^b F'(t)dt$$

$$Function\ Value = F(b) = F(a) + \int_a^b F'(t)dt$$

$$Function\ F(x) = F(a) + \int_a^x F'(t)dt\ (IVP)$$

ii. 物理衍生问题 Physical Derivation Problem

$$Displacement\ during\ [t1,t2] = \int_{t1}^{t2} v(t)dt$$

$$Distance\ Traveled\ during\ [t1,t2] = \int_{t1}^{t2} |v(t)|dt$$

$$Position\ s(b) = s(a) + \int_a^b v(t)dt$$

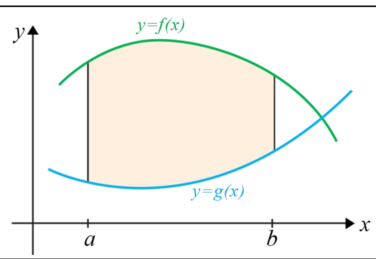
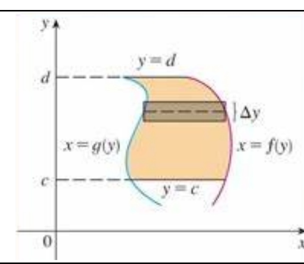
$$Position\ Function\ s(x) = s(a) + \int_a^x v(t)dt$$

18. 应用三：弧长求解 Application 3: Arc Length

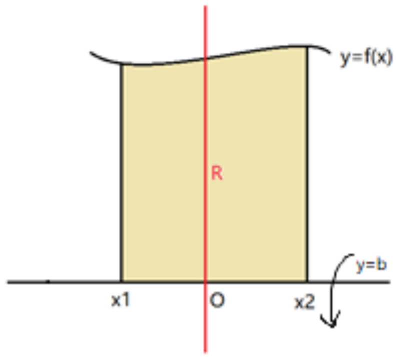
Let the function $y = f(x)$ represent a smooth curve on $[a,b]$

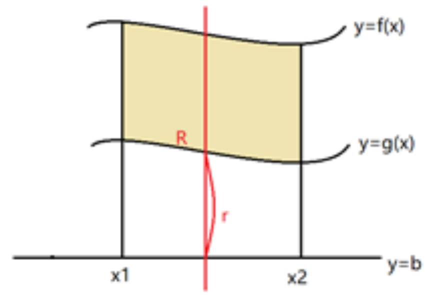
$$S = \int_a^b \sqrt{1+[f'(x)]^2}dx \quad S = \int_c^d \sqrt{1+[g'(y)]^2}dy$$

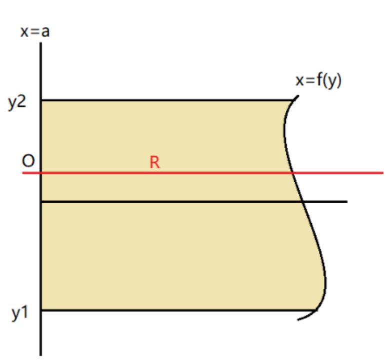
19. 应用四: 两条曲线之间区域面积 Application 4: Area of a Region between Two Curves

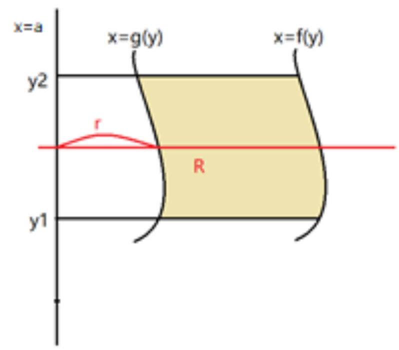
<table>
<tr><td>上下型，对 X 积分</td><td>右左型，对 Y 积分</td></tr>
<tr><td>
</td><td>
</td></tr>
<tr><td>$A = \int_a^b [f(x) - g(x)]dx$</td><td>$B = \int_c^d [f(y) - g(y)]dy$</td></tr>
<tr><td colspan="2">面积是对高的积分</td></tr>
</table>

20. 应用五: 圆盘体积求解 Application 5: The Volume of a Disk

<table>
<tr><th></th><th>Disk πR^2</th><th>Washer $\pi(R^2 - r^2)$</th></tr>
<tr><td>Horizontal
y=b
对 X 积分</td><td>

$\int_{x1}^{x2} \pi(f(x) - b)^2 dx$</td><td>

$\int_{x1}^{x2} \pi[(f(x) - b)^2 - (g(x) - b)^2 dx$</td></tr>
<tr><td>Vertical
x=a
对 Y 积分</td><td>

$\int_{y1}^{y2} \pi(f(y) - a)^2 dy$</td><td>

$\int_{y1}^{y2} \pi[(f(y) - a)^2 - (g(y) - a)^2 dy$</td></tr>
<tr><td colspan="3">体积是对截面的积分</td></tr>
</table>

21. 应用六: 具有已知横截面的固体 Application 6: Solids with Known Cross Section

a. 静态法求体积 Static Method

$$Volume = \int_a^b A(x)dx(\text{垂直于 X 轴}) \quad Volume = \int_a^b A(y)dy\,(\text{垂直与 Y 轴})$$

b. 求解步骤 Steps

i. Step 1: 确定与x/y轴垂直。

ii. Step 2: 画 Base 图，找出 Cross Section。

iii. Step 3: 求出 Cross Section 的边长$h(x)$（一般为两个函数相减）。

iv. Step 4: 利用求面积公式求出$A(x)$（一般为正方形，半圆，正三角形，等腰直角三角形）。

- 正方形面积$A = a^2$

- 等边三角形$A = \frac{\sqrt{3}}{4}a^2$
- 等腰直角三角形$A = \frac{1}{4}a^2$

v. Step 5: 确定积分区间。

vi. Step 6: 使用静态法求体积公式求解。

22. 反常积分 Improper Integrals

a. 定义 Definition: 为不定式，分母为 0 分子不为 0，使用 FTC 代入积分区间到被积函数后其不存在。

b. 垂直无界型反常积分 Vertical Unbounded Improper Integrals

i. f is continuous on the $[a, b]$ and $|f(x)| \to \infty, x \to b^-$

$$\int_a^b f(x)dx = \lim_{R \to b^-} \int_a^R f(x)dx$$

ii. f is continuous on the (a, b)

$$\int_a^b f(x)dx = \lim_{R \to a^+} \int_R^b f(x)dx$$

iii. Vertical asymptotes between $[a, b]$

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

$$= \lim_{R \to c^-} \int_a^R f(x)dx + \lim_{R \to a^+} \int_R^b f(x)dx$$

c. 水平无界型反常积分 Horizontal Unbounded Improper Integrals

$$\int_a^\infty f(x)dx = \lim_{R \to \infty} \int_a^R f(x)dx$$

$$\int_{-\infty}^a f(x)dx = \lim_{R \to -\infty} \int_R^a f(x)dx$$

d. 收敛和发散 Converges and Diverges

i. Limit exist, converges.

ii. Limit DNE, diverges.

第四单元 无穷级数 Chapter 4 Infinite Series

数列的极限与无穷级数 Limit of the Sequence and Infinite Series

1. 数列 Sequence
   a. 数列的定义 Definition of Sequence: Let L be a real number. The limit of a sequence $\{a_n\}$ is L.

   $$\lim_{n\to\infty} a_n = L$$

   Limit L of a sequence exists, converges; limit L of a sequence DNE, diverges.

   b. 数列极限的算法 How to Calculate the Limit of a Sequence
      i. 化为函数求解 Solve as A Function: 把数列函数化，之后使用函数的极限求解方法求解。

      $$\lim_{n\to\infty} a_n \to \lim_{n\to\infty} f(x) = L$$

      ii. 数列的三明治定理 Squeeze Theorem for Sequences

      If $\lim\limits_{n\to\infty} a_n = L = \lim\limits_{n\to\infty} b_n$ $(a_n \le c_n \le b_n)$, then $\lim\limits_{n\to\infty} c_n = L$.

2. 无穷级数 Infinite Series
   a. 无穷级数的定义 Definition of Infinite Series: 无穷数列的无穷项和。

   $$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots + a_n + \cdots$$

   b. 部分和数列 Sequence of Partial Sum
      i. 部分和数列 Sequence of Partial Sum：将级数中的前 n 项（一部分项）拿出来，组成一个新的数列。
      ii. 部分和数列判断级数收敛性 Sequence of Partial Sum to Determine Series Convergence: If this sequence of practical sums converges, the series is said to converge. 部分和数列与无穷级数同收敛同发散
      iii. 发散级数第 n 项判别法 The nth-Term Test for Divergence

      If $\lim\limits_{n\to\infty} a_n \neq 0$, then $\sum_{n=1}^{\infty} a_n$ diverge. If $\lim\limits_{n\to\infty} a_n = 0$, then no conclusion.

3. 绝对收敛与条件收敛 Absolute Convergence and Conditional Convergence
   a. 绝对收敛 Absolute Convergence
      i. $\sum a_n$ is absolutely convergent when $\sum |a_n|$ converges.
      ii. 加了绝对值后仍收敛为绝对收敛 Absolute Convergence。
   b. 条件收敛 Conditional Convergence
      i. $\sum a_n$ is conditional convergent when $\sum a_n$ converges but $\sum |a_n|$ diverges.

ii. 加了绝对值后变发散为条件收敛 Conditional Convergence。

四类重要级数及其判别法 Four Types of Important Series and Their Test

4. 几何级数 Geometric Series
   a. 特征 Characteristics: 首项 First Term 为 a，公比 Common Ratio 为 r ($r \neq 0$)

$$\sum_{n=0}^{\infty} ar^n = a + ar + ar^2 + \cdots + ar^n + \cdots \quad a \neq 0$$

   b. 几何级数判别法 The Geometric Series Test

   If $|r| < 1$, then $\sum_{n=1}^{\infty} ar^{n-1}$ converge to $\frac{a}{1-r}$; If $|r| \geq 1$, then $\sum_{n=1}^{\infty} ar^{n-1}$ diverge.

5. 伸缩级数 Telescoping Series
   a. 特征 Characteristics

$$(b_1 - b_2) + (b_2 - b_3) + (b_3 - b_4) + \cdots + (b_n - b_{n+1}) = b_1 - b_{n+1}$$

   b. 使用部分和数列判断伸缩级数收敛性 Using Sequence of Partial Sum to Determine Convergence of Telescoping Series

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} (b_1 - b_{n+1}) = b_1 - \lim_{n\to\infty} b_{n+1}$$

6. 超调和级数/P 级数 P-Series
   a. 特征 Characteristics

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \cdots + \frac{1}{n^p} + \cdots (p\ is\ a\ positive\ constant)$$

   b. 调和级数 Harmonic Series: 当 P=1 的时候的 P 级数。

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n} + \cdots (diverge)$$

   c. P 级数判别法 P-Series Test

   If $p > 1$, then $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converge; If $0 < p \leq 1$, then $\sum_{n=1}^{\infty} \frac{1}{n^p}$ diverge.

7. 交错级数 Alternating Series
   a. 特征 Characteristics: 符号数列 Symbolic Sequence + a_n

$$\sum_{n=1}^{\infty} (-1)^n a_n \quad \& \quad \sum_{n=1}^{\infty} (-1)^{n+1} a_n$$

   b. 交错级数判别法 Alternating Series Test

   Alternating series converge when two conditions listed below are met

- $\lim_{n\to\infty} a_n = 0$ and $a_{n+1} \le a_n \ for\ all\ n$
- If $\lim_{n\to\infty} a_n \ne 0$, by n-th term test, the series diverge.

正项级数的四大判别法 Four Tests of Nonnegative Series

8. 正项级数 Nonnegative Series
   a. 正项级数的定义 Definition of Nonnegative Series: If $a_n \ge 0$ for all n, then the series is called a nonnegative series.
   b. 正项级数收敛的充分必要条件 The Necessary and Sufficient Condition for Convergence of A Nonnegative Series: 其部分和数列有上界。
9. 积分判别法 The Integral Test
   a. 积分判别法内容 The Content of The Integral Test

      If f is positive, continuous, and decreasing for $x \ge 1$ and $a_n = f(n)$,

      then $\sum_{n=1}^{\infty} a_n$ and $\int_1^{\infty} f(x)dx$ either both converge and diverge.
   b. 做题步骤 Steps for Application
      i. Step 1: 级数函数化 $a_n = f(n) \to y = f(x)$
      ii. Step 2: 确定$f(x)$特征（正项 Positive，连续 Continuous，递减 Decreasing for $x \ge 1$）
      iii. Step 3: 对函数进行反常积分 Improper Integral。
         - 如果反常积分 Converge 则 Series Converge。
         - 如果反常积分 Diverge 则 Series Diverge。

$$\sum a_n \to \int_1^{\infty} f(x)dx = \lim_{R\to\infty} \int_1^{R} f(x)dx$$

10. 比值判别法 The Ratio Test

    $\sum a_n$ be a series with nonzero terms.
    - Series $\sum a_n$ converges absolutely when $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1$
    - Series $\sum a_n$ diverges when $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| > 1 \ or = \infty$
    - Ratio test is inconducive when $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1$
11. 根值判别法 The Root Test

$$\sum a_n \quad \lim_{n\to\infty} \sqrt[n]{|a_n|} = L$$

- $L < 1$ converges absolutely.
- $L > 1$ $(= \infty)$ diverges.
- $L = 1$ inconclusive.

12. 比较判别法 The Comparison Test
    a. 直接比较判别法 Direct Comparison Test

$$\text{Let } 0 < a_n \le b_n \text{ for all n}$$

$$\text{If } \sum_{n=1}^{\infty} b_n \text{ converges, } \sum_{n=1}^{\infty} a_n \text{ converges.}$$

$$\text{If } \sum_{n=1}^{\infty} a_n \text{ diverges, } \sum_{n=1}^{\infty} b_n \text{ diverges.}$$

大 Converge 小 Converge；小 Diverge 大 Diverge。

    b. 极限比较判别法 Limit Comparison Test

$$\text{If } a_n > 0, b_n > 0 \text{ and } \lim_{n\to\infty} \frac{a_n}{b_n} = L > 0, \text{ then}$$

$$\sum_{n=1}^{\infty} a_n \text{ and } \sum_{n=1}^{\infty} b_n \text{ either both converge, or both diverge.}$$

    c. 如何找出比较判别法中的对比级数 How to Find Comparison Series in the Comparison Test
        i. Step 1: 先把分母的括号去掉
        ii. Step 2: 确定分母分子
            - 如果分子为 1，保留分母最快的即可。

$$\frac{1}{3n^2 - 4n + 5} \to \frac{1}{n^2} \quad \frac{1}{\sqrt{3n-2}} \to \frac{1}{\sqrt{n}}$$

            - 如果分子不为 1，保留分子分母最快后，

$$\text{计算 } \frac{\text{新分母}}{\text{新分子}} = \text{Comparison Series 的分母}(\text{分子为 } 1)$$

幂级数、泰勒级数和麦克劳林级数 Power Series, Taylor Series and Maclaurin Series

13. 幂级数 Power Series
    a. 幂级数的定义 Definition of Power Series
        i. 函数项级数 Series with Function Terms: 幂函数是最基本的函数项级数，其每一项都是函数。
        ii. 表达式 Expression
            - A power series about $x = a$ is a series of the form

$$\sum_{n=0}^{\infty} a_n (x-a)^n = a_0 + a_1(x-a) + \cdots + a_n(x-a)^n + \cdots$$

- A power series about $x = 0$ is a series of the form

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots$$

b. 收敛半径与收敛区间 The Radius and Interval of Convergence

i. For a power series centered at c, precisely one of the following is true.

- The series converges only at c.
- $\sum_{n=0}^{\infty} a_n (x-a)^n$ exists a real number R>0 such that the series converges absolutely for $|x-c| < R$ and diverges for $|x-a| > R$.
- The series converges absolutely for all x.

ii. 用比值判别法判断幂级数收敛半径和收敛区间 Determining The Radius and Interval of Convergence of a Power Series by the Ratio Test

- 求解收敛区间 Solving for the Interval of Convergence

$$\lim_{n \to \infty} |\frac{a_{n+1}}{a_n}| < 1$$

在确认收敛区间后需要对端点专门讨论:

Step 1: 带入端点到原级数。

Step 2: 用级数判别法判断级数的收敛与发散。

Step 3: 若收敛，则纳入收敛区间内。

- 求解收敛半径 Solving for the Radius of Convergence

$$\lim_{n \to \infty} \left|\frac{a_{n+1}}{a_n}\right| \quad |x| < R$$

用 Ratio Test 解不等式后，化为$|x| < R$或$|x + b| < R$的形式，R 为其收敛半径。

c. 特殊的幂函数: 几何幂级数 Special Power Function: Geometric Power Series

i. 表达式 Expression

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$$

ii. 求解收敛区间: 公比 r 小于 1 的对应区间。

iii. Geometric power series converge to $a/1 - r$（首项/1-公比）

14. 泰勒级数和麦克劳林级数 Taylor and Maclaurin Series

a. 泰勒级数 Taylor Series

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \cdots$$

b. 麦克劳林级数 Maclaurin Series

$$f(x)=f(0)+f'(0)x+\frac{f''(0)}{2!}x^2+\cdots+\frac{f^{(n)}(0)}{n!}x^n+\cdots$$

麦克劳林级数是泰勒级数在$x=0$处的特例。

c. 常用的麦克劳林级数 Common Maclaurin Series

$$\frac{1}{1+x}=\sum_{n=0}^{\infty}(-1)^n x^n=1-x+x^2-x^3+\cdots+(-1)^n x^n+\cdots,|x|<\infty$$

$$e^x=\sum_{n=0}^{\infty}\frac{x^n}{n!}=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots+\frac{x^n}{n!}+\cdots,|x|<\infty$$

$$\sin x=\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!}=x-\frac{x^3}{3!}+\frac{x^5}{5!}-\cdots+(-1)^n\frac{x^{2n+1}}{(2n+1)!}+\cdots,|x|<\infty$$

$$\cos x=\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!}=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\cdots+(-1)^n\frac{x^{2n}}{(2n)!}+\cdots,|x|<\infty$$

遇到上述级数的变形级数时，直接对上述级数展开式变形

- 类型一：新级数对x变换，比如e^{2x}: 直接将新的x的线性表达带入原级数展开式中x的部分

$$e^x=\sum_{n=0}^{\infty}\frac{x^n}{n!}\ \rightarrow\ e^{2x}=\sum_{n=0}^{\infty}\frac{2x^n}{n!}$$

- 类型二：新级数对整个表达式变换，比如xe^x: 直接把该变化应用于原级数展开式的每一项

$$e^x=\sum_{n=0}^{\infty}\frac{x^n}{n!}\ \rightarrow\ xe^x=\sum_{n=0}^{\infty}x\frac{x^n}{n!}$$

15. 泰勒多项式 Taylor Polynomial

$$P_n(x)=f(a)+f'(a)(x-a)+\frac{f''(a)}{2!}(x-a)^2+\cdots+\frac{f^{(n)}(a)}{n!}(x-a)^n$$

泰勒多项式用于近似，微分应用的线性近似 Linear Approximation 等于一阶泰勒多项式；相比于泰勒级数泰勒多项式是有限项和。

16. 泰勒级数的导数和积分 Derivatives and Integrals of Taylor Series

$$f(x)=\sum_{n=0}^{\infty}a_n(x-c)^n\qquad f'(x)=\sum_{n=1}^{\infty}na_n(x-c)^{n-1}$$

$$\int f(x)dx = C + \sum_{n=1}^{\infty} a_n \frac{(x-c)^{n+1}}{n+1}$$

17. 幂级数变换法则 Operations with Power Series

    Let $f(x) = \sum_{n=0}^{\infty} a_n\, x^n$ and $g(x) = \sum_{n=0}^{\infty} b_n\, x^n$

    a. 线性: $f(kx) = \sum a_n k^n x^n$
    b. 幂次: $f(x^N) = \sum a_n x^{nN}$
    c. 交集 → 求定义域: $f(x) \pm g(x) = \sum (a_n \pm b_n) x^n$

误差界 Error Bound

18. 交错级数的误差界 Alternating Series Error Bound
    a. 表达式 Expression

    $$|R_n| = |S - S_n| \leq a_{n+1}$$

    b. 表达式含义 Meaning of the Expression
        i. 真实值减去估计值的绝对值小于等于组成级数的下一项。
        ii. $a_{n+1} \leq a_n$和$\lim_{n\to\infty} a_n = 0$的交错级数有效。

19. 拉格朗日误差界 Lagrange Error Bound
    a. 表达式 Expression

    $$R_n(x) < \max \left| \frac{f^{(n+1)}(c)(x-a)^{n+1}}{(n+1)!} \right|$$

    b. 表达式含义 Meaning of the Expression: 对函数值$f(x)$用其泰勒多项式逼近的精确程度，可以使用拉格朗日误差界度量。

第五单元 参数方程、向量函数与极坐标 Chapter 5 Parametric Equations, Vector Functions and Polar Coordinates

微积分、参数方程与平面中的向量 Calculus, Parametric Equations, Vectors in the Plane

1. 微积分与参数方程 Calculus and Parametric Equations
   a. 导数的参数形式 Parametric Form of Derivative
   $$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} \quad \frac{dx}{dt} \neq 0$$
   b. 参数形式的弧长 Arc Length in Parametric Form
   $$S = \int_a^b \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2}\, dt$$
2. 平面中的向量 Vectors in the Plane
   a. 向量函数 Vector Function: 向量函数是参数方程的另一种表示。
   $$x = x(t) \ \ y = y(t) \ \rightarrow \ \overrightarrow{s(t)} =< x(t), y(t) >$$
   b. 位置、速度与加速度向量 Position, Velocity, and Acceleration Vectors
      i. Position Vector: $\overrightarrow{\mathrm{s(t)}} =< \mathrm{x(t), y(t)} >$
      ii. Velocity Vector: $\overrightarrow{\mathrm{v(t)}} =< \mathrm{x'^{(t)}, y'(t)} >$
      iii. Acceleration Vector: $\overrightarrow{\mathrm{a(t)}} =< \mathrm{x''^{(t)}, y''(t)} >$
      iv. Speed $= \left|\overrightarrow{\mathrm{v(t)}}\right| = \sqrt{\left(\mathrm{x'(t)}\right)^2 + \left(\mathrm{y'(t)}\right)^2}$
      v. Distance $= \int_a^b \mathrm{speed\, dt}$

微积分与极坐标 Calculus and Polar Coordinates

3. 极坐标与平面直角坐标的转换 Converting Polar and Cartesian Coordinates
   $$x = rcos\theta \quad y = rsin\theta$$
4. 斜率与切线 Slope and Tangent Lines
   a. 极坐标中的斜率与切线 Slope and Tangent Lines in Polar Coordinates
   $$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} \quad \frac{dx}{d\theta} \neq 0$$
   b. 水平切线 Horizontal Tangent Line: 分子为 0，分母不为 0。

$$Horizontal\ Tangent\ Line: \frac{dy}{dx} = 0 = slope$$

c. 竖直切线 Vertical Tangent Line: 分子不为 0 为常数，分母为 0。

$$Vertical\ Tangent\ Line: \frac{dy}{dx} = \infty = slope$$

5. 极坐标形式的弧长 Arc Length in Polar Form

$$S = \int_{\alpha}^{\beta} \sqrt{[f(\theta)]^2 + [f'(\theta)]^2}\, d\theta = \int_{\alpha}^{\beta} \sqrt{r^2 + (\frac{dr}{d\theta})^2}\, d\theta$$

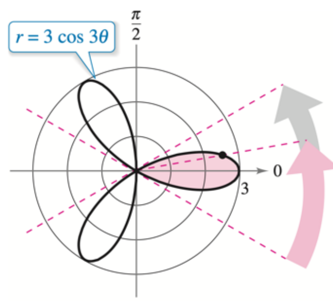
6. 极坐标图形的面积 Area of a Polar Region

a. 面积计算公式 Formula of Area

$$A = \frac{1}{2}\int_{\alpha}^{\beta} [f(\theta)]^2\, d\theta = \frac{1}{2}\int_{\alpha}^{\beta} r^2\, d\theta$$

b. 类型一：极坐标区域面积求解 Solving the Area of Polar Coordinates

i. 案例 Example: Find the area of one petal of the rose curve $r = 3cos3\theta$.



ii. Step 1: 令原极坐标方程等于 0，解三角方程，在无数解中挑选最简单并且与图中终边对应的角。

$$r = 3cos3\theta = 0$$

$$3\theta = \frac{\pi}{2} \quad \theta = \frac{\pi}{6}$$

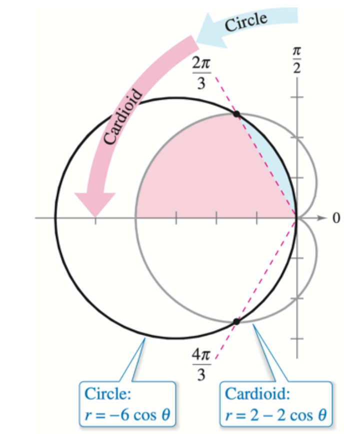
iii. Step 2: 使用公式计算。

$$A = \frac{1}{2}\int_{\alpha}^{\beta} r^2 d\theta = \frac{1}{2}\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} (3\cos 3\theta)^2 d\theta$$

$$= \frac{9}{4}[\theta + \frac{\sin 6\theta}{6}]_{-\frac{\pi}{6}}^{\frac{\pi}{6}} = \frac{3\pi}{4}$$

c. 类型二：求两条曲线之间区域的面积 Finding the Area of a Region Between Two Curves

i. 实例：Find the area of the region common to the two regions bounded by the curves. $r = -6cos\theta, r = 2 - 2cos\theta$



ii. Step 1: 连接交点与圆心，分割面积，求出题目给出的两个极坐标方程的交点

$$-6cos\theta = 2 - 2cos\theta$$

$$\theta = \frac{2\pi}{3}, \frac{4\pi}{3}$$

iii. Step 2: 逆时针划定上下限，带公式求解 $A = \frac{1}{2}\int_{\text{逆时针起点}}^{\text{逆时针终点}} [r]^2\, d\theta$

$$\frac{A}{2} = \overbrace{\frac{1}{2}\int_{\pi/2}^{2\pi/3} (-6\cos\theta)^2\, d\theta}^{\text{Region between circle and radial line } \theta = 2\pi/3} + \overbrace{\frac{1}{2}\int_{2\pi/3}^{\pi} (2 - 2\cos\theta)^2\, d\theta}^{\text{Region between cardioid and radial lines } \theta = 2\pi/3 \text{ and } \theta = \pi}$$

附录 Appendix

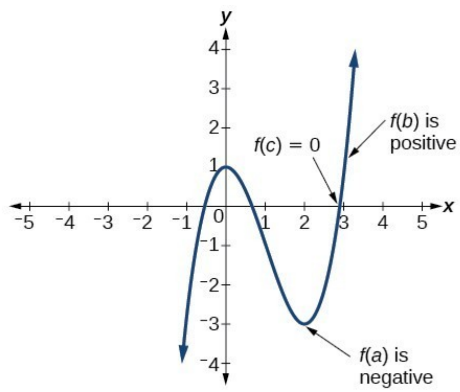
重要定理 Big Theorem

1. 三明治定理/夹逼定理 Squeeze Theorem

   If $h(x) \leq f(x) \leq g(x)$ and $\lim_{x\to c} h(x) = \lim_{x\to c} g(x) = L$, thus $\lim_{x\to c} f(x) = L$.

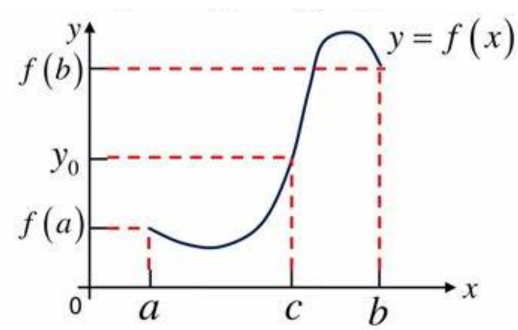
2. 零点存在性定理 Existence Theorem of Zero Points

   如果函数$y = f(x)$在区间$[a,b]$上的图象是连续不断的一条曲线，并且有$f(a) \cdot f(b) < 0$，那么，函数$y = f(x)$在区间(a,b)内有零点，即至少存在一个$c \in (a,b)$，使得$f(c) = 0$。（函数的两端一正一负，则必有零点出现。）



3. 介值定理 Intermediate Value Theorem (IVT)

   If f is continuous on the closed interval $[a,b]$, $f(a) \neq f(b)$, and k is any number between $f(a)$ and $f(b)$, then there is at least one number c in $[a,b]$ such that $f(c) = k$.（在连续函数的一个区间内的函数值肯定介于最大值和最小值之间。）



4. 极值定律 Extreme Value Theorem (EVT)

   If f is continuous on a closed interval $[a,b]$, then f has both a minimum and a maximum on the interval.

5. 罗尔中值定理 Rolle's Theorem

   Let f be continuous on the closed interval $[a,b]$ and differentiable on the open interval (a,b). If $f(a) = f(b)$, then there is at least one number c in (a,b) such that $f'(c) = 0$.

6. 微分中值定理/拉格朗日中值定理 Mean Value Theorem (MVT)
$f(x)$在$[a,b]$上连续并且在(a,b)上可导，则c属于(a,b)。

$$f'(c)=\frac{f(b)-f(a)}{b-a}$$

7. 微积分基本定理 Fundamental Theorem of Calculus (FTC)

$$\int_a^b f(x)dx=F(b)-F(a)=F(x)|_a^b \quad F'(x)=f(x), F(x)=\int f(x)dx$$

8. 积分中值定理 Mean Value Theorems for Definite Integrals (MVT for Integrals)
If f is continuous on the closed interval $[a,b]$, then there exists a number c in the closed interval $[a,b]$, such that

$$\int_a^b f(x)dx=f(c)(b-a)$$

9. 数列夹逼定理 Squeeze Theorem for Sequences
If $\lim_{n\to\infty} a_n=L=\lim_{n\to\infty} b_n$ $(a_n\le c_n\le b_n)$, then $\lim_{n\to\infty} c_n=L$.

10. 绝对值定理 Absolute Value Theorem
For the sequence $\{a_n\}$, if $\lim_{n\to\infty}|a_n|=0$, then $\lim_{n\to\infty} a_n=0$.

三角函数 Trigonometric Function

1. 特殊的三角函数值 Special Trigonometric Function Values

	0	$\frac{\pi}{6}$ 30°	$\frac{\pi}{4}$ 45°	$\frac{\pi}{3}$ 60°	$\frac{\pi}{2}$ 90°	π 180°
$\sin x$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	0
$\cos x$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	-1
$\tan x$	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	DNE	0

2. 三角函数的基本变形 Basic Transformation of Trigonometric Function

$$\sin(-\alpha)=-\sin\alpha \quad \cos(-\alpha)=\cos\alpha \quad \tan(-\alpha)=-\tan\alpha$$

$$\sec\alpha=\frac{1}{\cos\alpha} \quad \csc\alpha=\frac{1}{\sin\alpha} \quad \cot\alpha=\frac{1}{\tan\alpha}$$

$$\sin^2\alpha+\cos^2\alpha=1 \quad 1+\tan^2\alpha=\sec^2\alpha \quad 1+\cot^2\alpha=\csc^2\alpha$$

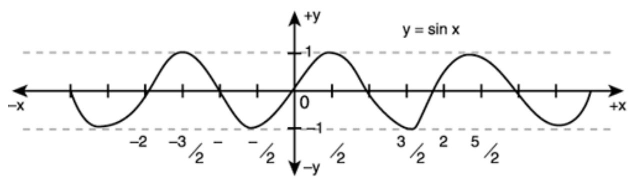
3. 二倍角公式 Double-Angle Formula

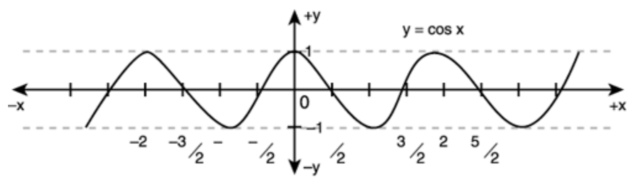
$$\sin 2\alpha = 2\sin\alpha\cos\alpha$$

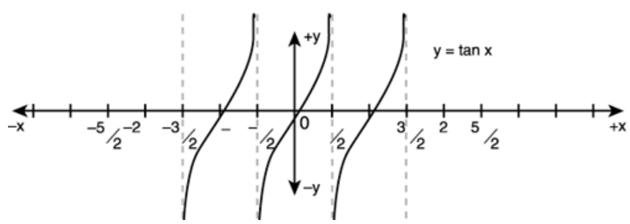
$$\cos 2\alpha = \cos^2\alpha - \sin^2\alpha = 2\cos^2\alpha - 1 = 1 - 2\sin^2\alpha$$

$$\tan 2\alpha = \frac{2\tan\alpha}{1-\tan^2\alpha}$$

4. 基本三角函数图形 Basic Trigonometric Function Graph







5. 三角函数周期公式 Trigonometric Function Period Formula

$$\sin(x+2k\pi) = sinx \quad \cos(x+2k\pi) = cosx \quad \tan(x+k\pi) = tanx$$

6. 单位圆 Unit Circle